Progressions for the Common Core State Standards in Mathematics (draft)

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Grade 8, High School, Functions*

Overview

Functions describe situations in which one quantity is determined by another. The area of a circle, for example, is a function of its radius. When describing relationships between quantities, the defining characteristic of a *function* is that the input value determines the output value or, equivalently, that the output value depends upon the input value.

The mathematical meaning of function is quite different from some common uses of the word, as in, "One function of the liver is to remove toxins from the body," or "The party will be held in the function room at the community center." The mathematical meaning of function is close, however, to some uses in everyday language. For example, a teacher might say, "Your grade in this class is a function of the effort you put into it." A doctor might say, "Some illnesses are a function of stress." Or a meteorologist might say, "After a volcano eruption, the path of the ash plume is a function of wind and weather." In these examples, the meaning of "function" is close to its mathematical meaning.

In some situations where two quantities are related, each can be viewed as a function of the other. For example, in the context of rectangles of fixed perimeter, the length can be viewed as depending upon the width or vice versa. In some of these cases, a problem context may suggest which one quantity to choose as the input variable.

Undergraduate mathematics may involve functions of more than one variable. The area of a rectangle, for example, can be viewed as a function of two variables: its width and length. But in high school mathematics the study of functions focuses primarily on real-valued functions of a single real variable, which is to say that both the input and output values are real numbers. One exception is in high school geometry, where geometric transformations are considered to be functions.• For example, a translation T, which moves the plane

• G-CO.2 . . . [D]escribe transformations as functions that take points in the plane as inputs and give other points as outputs. . . .

*Advanced material, corresponding to (+) standards, is indicated by plus signs in the left margin.

3 units to the right and 2 units up might be represented by $T : (x, y) \mapsto (x + 3, y + 2)$.

A trickle of pattern standards in Grades 4 and 5 begins the preparation for functions.[4.OA.5, 5.OA.3] Note that in both these standards a rule is explicitly given. Traditional pattern activities, where students are asked to continue a pattern through observation, are not a mathematical topic, and do not appear in the Standards in their own right.[1]

The Grade 4–5 pattern standards expand to a full progression on Ratios and Proportional Relationships in Grades 6–7, and then the notion of a function is introduced in Grade 8.

Before they learn the term "function," students begin to gain experience with functions in elementary grades. In Kindergarten, they use patterns with numbers such as the $5 + n$ pattern to learn particular additions and subtractions.

Sequences and functions Patterns are sequences, and sequences are functions with a domain consisting of whole numbers. However, in many elementary patterning activities, the input values are not given explicitly. In high school, students learn to use an index to indicate which term is being discussed. Erica handles this issue in the example in the margin by deciding that the term 2 would correspond to an index value of 1. Then the terms 4, 6, and 8 would correspond to input values of 2, 3, and 4, respectively. Erica could have decided that the term 2 would correspond to a different index value, such as 0. The resulting formula would have been different, but the (unindexed) sequence would have been the same.

Functions and modeling In modeling situations, knowledge of the context and statistics are sometimes used together to find algebraic expressions that best fit an observed relationship between quantities. Then the algebraic expressions can be used to interpolate (i.e., approximate or predict function values between and among the collected data values) and to extrapolate (i.e., to approximate or predict function values beyond the collected data values). One must always ask whether such approximations are reasonable in the context.

In school mathematics, functional relationships are often given by algebraic expressions. For example, $f(n) = n^2$ for $n \geqslant 1$ gives the nth square number. But in many modeling situations, such as the temperature at Boston's Logan Airport as a function of time, algebraic expressions are generally not suitable.

Functions and Algebra See the Algebra Progression for a discussion of the connection and distinctions between functions, on the one hand, and algebra and equation solving, on the other. Perhaps the

[1] This does not exclude activities where patterns are used to support other standards, as long as the case can be made that they do so.

4.OA.5 Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

5.OA.3 Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

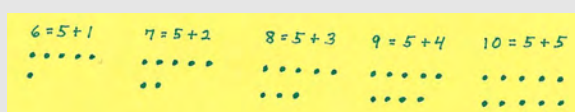
The problem with patterns

Students are asked to continue the pattern 2, 4, 6, 8, ... Here are some legitimate responses:

- Cody: I am thinking of a "plus 2 pattern," so it continues 10, 12, 14, 16, ...
- Ali: I am thinking of a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, ...
- Suri: I am thinking of the units digit in the multiples of 2, so it continues 0, 2, 4, 6, 8, 0, 2, ...
- Erica: If $g(n)$ is any polynomial, then $f(n) = 2n + (n-1)(n-2)(n-3)(n-4)g(n)$ describes a continuation of this sequence.
- Zach: I am thinking of that high school cheer, "Who do we appreciate?"

Because the task provides no structure, all of these answers must be considered correct. Without any structure, continuing the pattern is simply speculation—a guessing game. Because there are infinitely many ways to continue a sequence, patterning problems should provide enough structure so that the sequence is well defined.

Experiences with functions before Grade 8



$f(n) = 5 + n$ *(pattern from Kindergarten, p. ??)*

$$1 \times 9 = 9$$
$$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$$
$$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$$

$f(n) = 9 \times n = 10 \times n - n$ *(pattern from grade 3, p. ??)*

feet	inches
0	0
1	12
2	24
3	

$f(t) = 12t$ *(foot and inch equivalences from grade 4, p. ??)*

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

$f(t) = \frac{3}{2}t$ *(proportional relationship from Grade 6, p. ??)*

most productive connection is that solving equations can be seen as finding the intersections of graphs of functions.[A-REI.11]

What to expect from this document The study of functions occupies a large part of a student's high school career, and this document does not treat in detail all of the material studied. Rather it gives some general guidance about ways to treat the material and ways to tie it together.

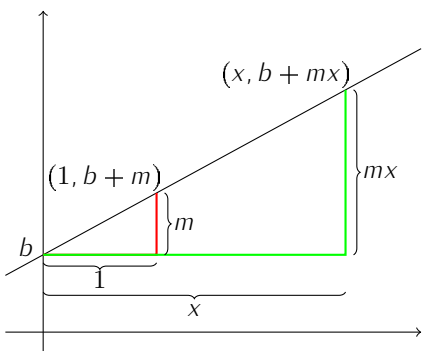
A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Grade 8

Define, evaluate, and compare functions Since the elementary grades, students have been describing patterns and expressing relationships between quantities. These ideas become semi-formal in Grade 8 with the introduction of the concept of *function*: a rule that assigns to each input exactly one output.[8.F.1] Formal language, such as domain and range, and function notation may be postponed until high school.

Building on experience with graphs and tables in Grades 6 and 7, students establish a routine of exploring functional relationships algebraically, graphically, numerically in tables, and through verbal descriptions.[8.F.2, MP1] And to develop flexibility in interpreting and translating among these various representations, students compare two functions represented in different ways, as illustrated by the task in the margin.

The main focus in Grade 8 is linear functions, those of the form $y = mx + b$, where m and b are constants.[8.F.3] The proof that $y = mx + b$ is also the equation of a line, and hence that the graph of a linear function is a line, is an important pieces of reasoning connecting algebra with geometry in Grade 8.[8.EE.6]



In the figure above, the red triangle is the "slope triangle" formed by the vertical intercept and the point on the line with x-coordinate equal to 1. The green triangle is formed from the intercept and a point with arbitrary x-coordinate. A dilation with center at the vertical intercept and scale factor x takes the red triangle to the green triangle, because it takes lines to parallel lines. Thus the green triangle is similar to the red triangle, and so the height of the green triangle is mx, and the coordinates of the general point on the triangle are $(x, b + mx)$. Which is to say that the point satisfies the equation $y = b + mx$.

Students learn to recognize linearity in a table: when constant differences between input values produce constant differences between output values. And they can use the constant rate of change appropriately in a verbal description of a context.

8.F.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.Function notation is not required in Grade 8.

8.F.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

MP1 "Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends"

Battery charging

Sam wants to take his MP3 player and his video game player on a car trip. An hour before they plan to leave, he realized that he forgot to charge the batteries last night. At that point, he plugged in both devices so they can charge as long as possible before they leave.
Sam knows that his MP3 player has 40% of its battery life left and that the battery charges by an additional 12 percentage points every 15 minutes.
His video game player is new, so Sam doesn't know how fast it is charging but he recorded the battery charge for the first 30 minutes after he plugged it in.

time charging	0	10	20	30
video game player batter charge	20	32	44	56

1. If Sam's family leaves as planned, what percent of the battery will be charged for each of the two devices when they leave?
2. How much time would Sam need to charge the battery 100% on both devices?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/641

8.F.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

8.EE.6 Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Use functions to model relationships between quantities When using functions to model a linear relationships between quantities, students learn to determine the rate of change of the function, which is the slope of its graph. They can read (or compute or approximate) the rate of change from a table or a graph, and they can interpret the rate of change in context.[8.F.4]

Graphs are ubiquitous in the study of functions, but it is important to distinguish a function from its graph. For example, a function does not have a slope but its graph can have a slope.•

Within the class of linear functions, students learn that some are proportional relationships and some are not. Functions of the form $y = mx + b$ are proportional relationships exactly when $b = 0$, so that y is proportional to x. Graphically, a linear function is a proportional relationship if its graph goes through the origin.

To understand relationships between quantities, it is often helpful to describe the relationships qualitatively, paying attention to the general shape of the graph without concern for specific numerical values.[8.F.5] The standard approach proceeds from left to right, describing what happens to the output as the input value increases. For example, pianist Chris Donnelly describes the relationship between creativity and structure via a graph.•

The qualitative description might be as follows: "As the input value (structure) increases, the output (creativity) increases quickly at first and gradually slowing down. As input (structure) continues to increase, the output (creativity) reaches a maximum and then starts decreasing, slowly at first, and gradually faster." Thus, from the graph alone, one can infer Donnelly's point that there is an optimal amount of structure that produces maximum creativity. With little structure or with too much structure, in contrast, creativity is low.

8.F.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

• The slope of a vertical line is undefined and the slope of a horizontal line is 0. Either of these cases might be considered "no slope." Thus, the phrase "no slope" should be avoided because it is imprecise and unclear.

8.F.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

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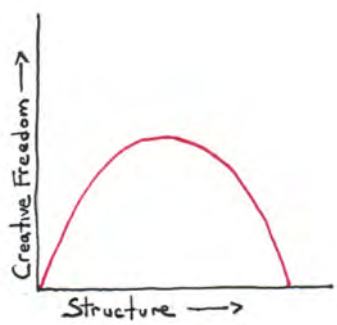


High School

The high school standards on functions are organized into four groups: Interpreting Functions (F-IF); Building Functions (F-BF); Linear, Quadratic and Exponential Models (F-LE); and Trigonometric Functions (F-TF). The organization of the first two groups under mathematical practices rather than types of function is an important aspect of the Standards: students should develop ways of thinking that are general and allow them to approach any type of function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary. For example, they should see linear and exponential functions as arising out of structurally similar growth principles; they should see quadratic, polynomial, and rational functions as belonging to the same system (helped along by the unified study in the Algebra category of Arithmetic with Polynomials and Rationl Expressions).

Interpreting Functions

Understand the concept of a function and use function notation Building on semi-formal notions of functions from Grade 8, students in high school begin to use formal notation and language for functions. Now the input/output relationship is a correspondence between two sets: the domain and the range.[F-IF.1] The domain is the set of input values, and the range is the set of output values. A key advantage of function notation is that the correspondence is built into the notation. For example, $f(5)$ is shorthand for "the output value of f when the input value is 5."

Students sometimes interpret the parentheses in function notation as indicating multiplication. Because they might have seen numerical expressions like $3(4)$, meaning 3 times 4, students can interpret $f(x)$ as f times x. This can lead to false generalizations of the distributive property, such replacing $f(x+3)$ with $f(x)+f(3)$. Work with interpreting function notation in terms of the graph of f can help students avoid this confusion with the symbols (see example in margin).

Although it is common to say "the function $f(x)$," the notation $f(x)$ refers to a single output value when the input value is x. To talk about the function as a whole, write f, or perhaps "the function f, where $f(x) = 3x+4$." The x is merely a placeholder, so $f(t) = 3t+4$ describes exactly the same function.

Later, students can make interpretations like those in the following table:

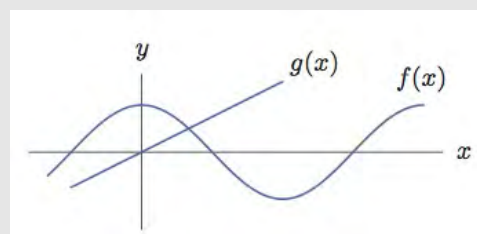
Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x
$f(b)-f(a)$	The change in output when the input changes from a to b

Notice that a common preoccupation of high school mathematics,

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Interpreting the graph

Use the graph (for example, by marking specific points) to illustrate the statements in (a)–(d). If possible, label the coordinates of any points you draw.



(a) $f(0) = 2$

(b) $f(3) = f(3) = f(9) = 0$

(c) $f(2) = g(2)$

(d) $g(x) > f(x)$ for $x > 2$

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/636.

The square root function

Since the equation $x^2 = 9$ has two solutions, $x = \pm 3$, students might think incorrectly that $\sqrt{9} = \pm 3$. However, if we want $\sqrt{x}$ to be a function of x, we need to choose one of these square roots. The square root function, $g(x) = \sqrt{x}$, is defined to be the positive square root of x for any positive x.

distinguishing function from relations, is not in the Standards. Time normally spent on exercises involving the vertical line test, or searching lists of ordered pairs to find two with the same x-coordinate and different y-coordinate, can be reallocated elsewhere. Indeed, the vertical line test is problematical, since it makes it difficult to discuss questions such as "is x a function of y" when presented with a graph of y against x (an important question for students thinking about inverse functions). The core question when investigating functions is: "Does each element of the domain corresponds to exactly one element in the range?" The margin shows a discussion of the square root function oriented around this question.

To promote fluency with function notation, students interpret function notation in contexts.[F-IF.2] For example, if h is a function that relates Kristin's height in inches to her age in years, then the statement $h(7) = 49$ means, "When Kristin was 7 years old, she was 49 inches tall." The value of $h(12)$ is the answer to "How tall was Kristin when she was 12 years old." And the solution of $h(x) = 60$ is the answer to "How old was Kristen when she was 60 inches tall?" See also the example in the margin.

Sometimes, especially in real-world contexts, there is no expression (or closed formula) for a function. In those cases, it is common to use a graph or a table of values to (partially) represent the function.

A *sequence* is a function whose domain is a subset of the integers.[F-IF.3] In fact, many patterns explored in grades K-8 can be considered sequences. For example, the sequence 4, 7, 10, 13, 16, ... might be described as a "plus 3 pattern" because terms are computed by adding 3 to the previous term. To show how the sequence can be considered a function, we need an *index* that indicates which term of the sequence we are talking about, and which serves as an input value to the function. Deciding that the 4 corresponds to an index value of 1, we make a table showing the correspondence, as in the margin. The sequence can be describe recursively by the rule $f(1) = 4$, $f(n+1) = f(n) + 3$ for $n \geqslant 2$. Notice that the recursive definition requires both a starting value and a rule for computing subsequent terms. The sequence can also be described with the closed formula $f(n) = 3n + 1$, for integers $n \geqslant 1$. Notice that the domain is included as part of the description. A graph of the sequence consists of discrete dots, because the specification does not indicate what happens "between the dots."

+ In advanced courses, students may use subscript notation for
+ sequences.

Interpret functions that arise in applications in terms of the context Functions are often described and understood in terms of their *behavior*:[F-IF.4] Over what input values is it increasing, decreasing, or constant? For what input values is the output value positive, negative, or 0? What happens to the output when the input value gets very large positively or negatively? Graphs become very useful

F-IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Cell Phones

Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$

(b) $f(a) = 20$

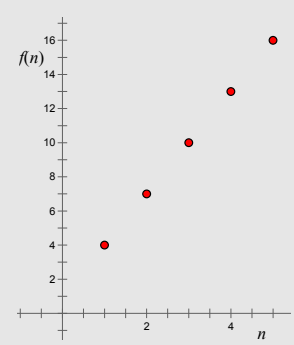
(c) $f(20) = b$

(d) $n = f(t)$

For solutions and more discussion of this task, go to illustrativemathematics.org/illustrations/634.

Sequences as functions

n	$f(n)$
1	4
2	7
3	10
4	13
5	16



F-IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

representations for understanding and comparing functions because these "behaviors" are often easy to see in the graphs of functions (see illustration in margin). Graphs and contexts are opportunities to talk about domain (for an illustration, go to illustrativemathematics.org/illustrations/631).[F-IF.5]

Graphs help us reason about rates of change of function. Students learned in Grade 8 that the *rate of change* of a linear function is equal to the slope of its graph. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For nonlinear functions, however, rates of change are not constant, and so we talk about average rates of change over an interval.[F-IF.6] For example, for the function $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line from $(2, 4)$ to $(5, 25)$ on the graph of g. And if g is interpreted as returning the area of a square of side x, then this calculation means that over this interval the area changes, on average, 7 square units for each unit increase in the side length of the square.

Analyze functions using different representations Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of function and its key features.[F-IF.7]

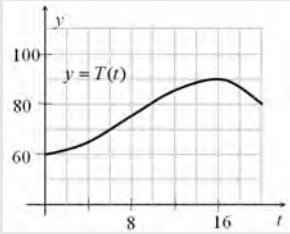
Within a family, the functions often have commonalities in the qualitative shapes of their graphs and in the kinds of features that are important for identifying functions more precisely within a family. This standard indicates which function families should be in students' repertoires, detailing which features are required for several key families. It is an overarching standard that covers the entire range of a student's high school experience; in this part of the progression we merely indicate some guidelines for how it should be treated.

First, linear and exponential functions (and to a lesser extent quadratic functions) receive extensive treatment and comparison in a dedicated group of standards, **Linear and Exponential Models**. Thus, those function families should receive the bulk of the attention related to this standard. Second, all students are expected to develop fluency with linear, quadratic, and exponential functions, including the ability to graph them by hand. Finally, in most of the other function families, students are expected to simple cases without technology, and more complex ones with technology.

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Warming and Cooling

The figure shows the graph of T, the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time t.



(a) Estimate $T(14)$.

(b) If $t = 0$ corresponds to midnight, interpret what we mean by $T(14)$ in words.

(c) Estimate the highest temperature during this period from the graph.

(d) When was the temperature decreasing?

(e) If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/639.

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a Graph linear and quadratic functions and show intercepts, maxima, and minima.

b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Consistent with the practice of looking for and making use of structure (MP1), students should also develop the practice of writing expressions for functions in ways that reveal the key features of the function.[F-IF.8]

Quadratic functions provide a rich playground for developing this ability, since the three principle forms for a quadratic expression (expanded, factored, and completed square) each give insight into different aspects of the function. However, there is a danger that working with these different forms becomes an exercise in picking numbers out of an expression. For example, students often arrive at college talking about "minus b over $2a$ method" for finding the vertex of the graph of a quadratic function. To avoid this problem it is useful to give students translation tasks such as the one in the margin, where they must read both the graphs and the expression and choose for themselves which parts of each correspond.[F-IF.9]

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

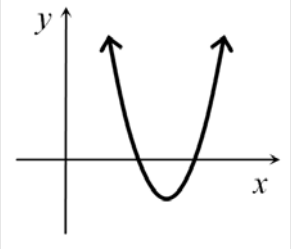
a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

b Use the properties of exponents to interpret expressions for exponential functions.

Which Equation?

Which of the following could be an expression for the function whose graph is shown below? Explain.

(a) $(x+12)^2+4$ (b) $-(x-2)^2-1$
(c) $(x+18)^2-40$ (d) $(x-10)^2-15$
(e) $-4(x+2)(x+3)$ (f) $(x+4)(x-6)$
(g) $(x-12)(-x+18)$ (h) $(20-x)(30-x)$



For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/640.

F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions

The previous group of standards focuses on interpreting functions given by expressions, graphs, or tables. The Building Functions group focuses on building functions to model relationships, and building new functions from existing functions.

Note: Composition and composition of a function and its inverse are among the plus standards. The following discussion describes in detail what is required for students to grasp these ideas securely. Because of the depth of this development, and because of the subtleties and pitfalls, it is strongly recommended that this content be included only in optional courses.

Build a function that models a relationship between two quantities This cluster of standards is very closely related to the algebra standard on writing equations in two variables.[A-CED.2] Indeed, that algebra standard might well be met by a curriculum in the same unit as this cluster. Although students will eventually study various families of functions, it is useful for them to have experiences of building functions from scratch, without the aid of a host of special recipes, by grappling with a concrete context for clues.[F-BF.1a] For example, in the Lake Algae task in the margin, parts (a)–(c) lead students through reasoning that allows them construct the function in part (d) directly. Students who try a more conventional approach in part (d) of fitting the general function $f(t) = ab^t$ to the situation might well get confused or replicate work already done.

The Algebra Progression discusses the difference between a function and an expression. Not all functions are given by expressions, and in many situations it is natural to use a function defined recursively. Calculating mortgage payment and drug dosages are typical cases where recursively defined functions are useful (see example in the margin).

Modeling contexts also provide a natural place for students to start building functions with simpler functions as components.[F-BF.1bc] Situations of cooling or heating involve functions which approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is $70°$ and a cup of tea made with boiling water at a temperature of $212°$, a student can express the function describing the temperature as a function of time using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, leading to a function of the form

$$T(t) = 70 + 142e^{-kt}.$$

Students might determine the constant k experimentally.

In contexts where change occurs at discrete intervals (such as payments of interest on a bank balance) or where the input vari-

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-BF.1a Write a function that describes a relationship between two quantities.

a Determine an explicit expression, a recursive process, or steps for calculation from a context.

Lake Algae

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

(a) When will the lake be covered half-way?

(b) On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

(c) On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

(d) Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/533.

Drug Dosage

A student strained her knee in an intramural volleyball game, and her doctor has prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220-milligram tablets every 8 hours for 10 days. Her kidneys filter 60% of this drug from her body every 8 hours. How much of the drug is in her system after 24 hours?

Task from High School Mathematics at Work: Essays and Examples for the Education of All Students (1998), National Academies Press. See http://www.nap.edu/openbook/0309063531/html/80.html for a discussion of the task.

F-BF.1 Write a function that describes a relationship between two quantities.

b Combine standard function types using arithmetic operations.

c (+) Compose functions.

able is a whole number (for example the number of a pattern in a sequence of patterns), the functions chosen will be sequences. In preparation for the deeper study of linear and exponential functions, students can study arithmetic sequences (which are linear functions) and geometric sequences (which are exponential functions).[F-BF.2] This is a good point at which to start making the distinction between additive and multiplicative changes.

Build new functions from existing functions With a basis of experiences in building specific functions from scratch, students start to develop a notion of naturally occurring families of functions that deserve particular attention. It is possible to harden the curriculum too soon around these families, before students have enough experience to get a feel for the effects of different parameters. Students can start getting that feel by playing around with the effect on the graph of simple transformations of the input and output variables.[F-BF.3] Quadratic and absolute value functions are good contexts for getting a sense of the effects of many of these transformations, but eventually students need to understand these ideas abstractly and be able to talk about them for any function f.

Students can get confused about the effect of transformations on the input variable, because the effect on the graph appears to be the opposite to the transformation on the variable. In part (b) of the task in the margin, asking students to talk through the positions of the points in terms of function values can help clear this confusion up.

The concepts of even and odd functions are useful for noticing symmetry. A function f is called an *even function* if $f(-x) = f(x)$ and an *odd function* if $f(-x) = -f(x)$. To understand the names of these concepts, consider that polynomial functions are even exactly when all terms are of even degree and odd exactly when all terms are of odd degree. With some grounding in polynomial functions, students can reason that lots of functions are neither even nor odd.

Students can show from the definitions that the sum of two even functions is even and the sum of two odd functions is odd, and they can interpret these results graphically.

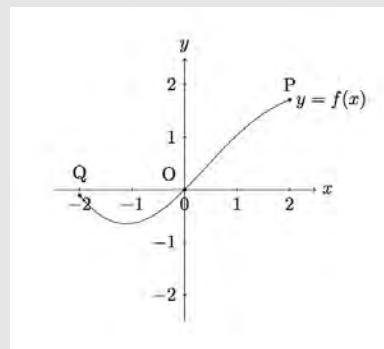
When it comes to inverse functions,[F-BF.4a] the expectations are modest, requiring only that students solve equations of the form $f(x) = c$. The point is to provide an informal sense of determining the input when the output is known. Much of this work can be done with specific values of c. Eventually, some generality is warranted. For example, if $f(x) = 2x^3$, then solving $f(x) = c$ leads to $x = (c/2)^{1/3}$, which is the general formula for finding an input from a specific output, c, for this function, f.

At this point, students need neither the notation nor the formal language of inverse functions, but only the idea of "going backwards" from output to input. This can be interpreted for a table and graph of the function under examination. Correspondences between equa-

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Transforming Functions

The figure shows the graph of a function f whose domain is the interval $-2 \leqslant x \leqslant 2$.



(a) In (i)–(iii), sketch the graph of the given function and compare with the graph of f. Explain what you see.

(i) $g(x) = f(x) + 2$

(ii) $h(x) = -f(x)$

(iii) $p(x) = f(x + 2)$

(b) The points labelled Q, O, P on the graph of f have coordinates

$$Q = (-2 - 0.509), \quad O = (0, -0.4), \quad P = (2, 1.309).$$

What are the coordinates of the points corresponding to P, O, Q on the graphs of g, h, and p?

For solutions and more discussion of this task, go to the Illustrative Mathematics website at illustrativemathematics.org/illustrations/742

F-BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

An Interesting Fact

Suppose f is a function with a domain of all real numbers. Define g and h as follows:

$$g(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad h(x) = \frac{f(x) - f(-x)}{2}$$

Then $f(x) = g(x) + h(x)$, g is even, and h is odd. (Students may use the definitions to verify these claims.) Thus, any function over the real numbers can be expressed as the sum of an even and an odd function.

F-BF.4a Find inverse functions.

a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

tions giving specific values of the functions, table entries, and point on the graph can be noted (MP1). And although not required in the standard, it is reasonable to include, for comparison, a few examples where the input cannot be uniquely determined from the output. For example, if $g(x) = x^2$, then $g(x) = 5$ has two solutions, $x = \pm\sqrt{5}$.

For advanced mathematics, some students will need a formal sense of inverse functions, which requires careful development. For example, as students begin formal study, they can easily believe that "inverse functions" are a new family of functions, similar to linear functions and exponential functions. To help students develop the instinct that "inverse" is a relationship between two functions, the recurring questions should be "What is the inverse of this function?" and "Does this function have an inverse?" The focus should be on "inverses of functions" rather than a new type of function.

Discussions of the language and notation for inverse functions can help to provide students a sense of what the adjective "inverse" means and mention that a function which has an inverse is known as an "invertible function."

The function $\mathcal{I}(x) = x$ is sometimes called the identity function because it assigns each number to itself. It behaves with respect to composition of functions the way the multiplicative identity, 1, behaves with multiplication of real numbers and the way that the identity matrix behaves with matrix multiplication. If f is any function (over the real numbers), this analogy can be expressed symbolically as $f \circ \mathcal{I} = f = \mathcal{I} \circ f$, and it can be verified as follows:

$$f \circ \mathcal{I}(x) = f(\mathcal{I}(x)) = f(x) \qquad \text{and} \qquad \mathcal{I} \circ f(x) = \mathcal{I}(f(x)) = f(x)$$

Suppose f denotes a function with an inverse whose domain is the real numbers and a is nonzero real number (which thus has a multiplicative inverse), and B is an invertible matrix. The following table compares the concept of inverse function with the concepts of multiplicative inverse and inverse matrix:

Sentence	Interpretation
$f^{-1} \circ f = \mathcal{I} = f \circ f^{-1}$	The composition of f^{-1} with f is the identity function
$a^{-1} \cdot a = 1 = a \cdot a^{-1}$	The product of a^{-1} and a is the multiplicative identity
$B^{-1} \cdot B = \mathcal{I} = B \cdot B^{-1}$	The product of B^{-1} and B is the identity matrix

In other words, where a^{-1} means the inverse of a with respect to multiplication, f^{-1} means the inverse of f with respect to function composition. Thus, when students interpret the notation $f^{-1}(x)$ incorrectly to mean $1/f(x)$, the guidance they need is that the meaning of the "exponent" in f^{-1} is about function composition, not about multiplication.

Students do not need to develop the abstract sense of identity and inverse detailed in the above table. Nonetheless, these perspectives can inform the language and conversation in the classroom as students verify by composition (in both directions) that given functions are inverses.[F-BF.4b]. Furthermore, students can continue to refine their informal "going backwards" notions, as they consider

A Joke

Teacher: Are these two functions inverses?
Student: Um, the first one is and the second one isn't.

What does this student misunderstand about inverse functions?

Notation Inconsistency

In the expression $\sin^2 x$, the exponent is about multiplication, but in $\sin^{-1} x$ the exponent is about function composition. Despite the similar look, these notations use superscripts in different ways. The 2 acts as an exponent but the -1 does not. Both notations, however, allow the expression to be written without the parentheses that would be needed otherwise.

F-BF.4b(+) Verify by composition that one function is the inverse of another.

inverses of functions given by graphs or tables.[F-BF.4c] In this work, students can gain a sense that "going backwards" interchanges the input and output and therefore the stereotypical roles of the letters x and y. And they can reason why the graph of $y = f^{-1}(x)$ will be the reflection across the line $y = x$ of the graph of $y = f(x)$.

Suppose $g(x) = (x-3)^2$. From the graph, it is easy to see that $g(x) = c$ will have two solutions for any $c > 0$. Thus, to create an invertible function,[F-BF.4d] we must restrict the domain of g so that every range value corresponds to exactly one domain value. One possibility is to restrict the domain of g to $x \geqslant 3$, as illustrated by the solid purple curve in the graph on the left.•

When solving $(x-3)^2 = c$, we get $x = 3 \pm \sqrt{c}$, illustrating that positive values of c will yield two solutions x for the unrestricted function. With the restriction, $3 - \sqrt{c}$ is not in the domain. Thus, $x = 3 + \sqrt{c}$, which corresponds to choosing the solid curve and ignoring the dotted portion. The inverse function, then, is $h(c) = 3 + \sqrt{c}$, for $c \geqslant 0$.

We check that h is the inverse of (restricted) g as follows:

$$g(h(x)) = g\left(3 + \sqrt{x}\right) = \left((3 + \sqrt{x}) - 3\right)^2 = (\sqrt{x})^2 = x, \quad x \geqslant 0$$

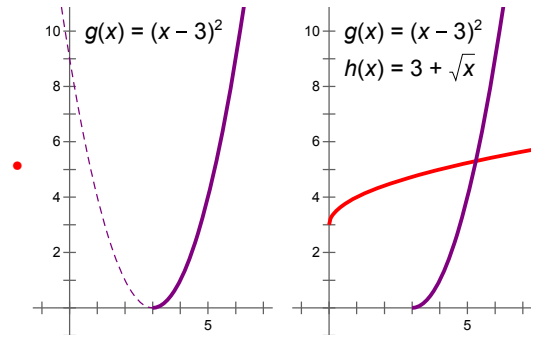
$$h(g(x)) = h\left((x-3)^2\right) = 3 + \sqrt{(x-3)^2} = 3 + (x-3) = x, \quad x \geqslant 3.$$

The first verification requires that $x \geqslant 0$ so that x is in the domain of h. The second verification requires that $x \geqslant 3$ so that x is in the domain of (restricted) g. And this is precisely what is required for $\sqrt{(x-3)^2}$ to simplify to $(x-3)$.• The rightmost graph shows that the graph of h can be seen as the reflection of the graph of g across the line $y = x$.

As detailed in the next section, logarithms for most students are merely shorthand for the solutions to exponential equations. Students in advanced classes, however, need to understand logarithms as functions—and as inverses of exponential functions.[F-BF.5] They should be able to explain identities such as $\log_b(b^x) = x$ and $b^{\log_b x} = x$ as well as the laws of logarithms, such as $\log(ab) = \log a + \log b$. In doing so, students can think of the logarithms as unknown exponents in expressions with base 10 (e.g. $\log a$ answers the question "Ten to the what equals a?") and use the properties of exponents,[N-RN.1] building on the understanding of exponents that began in Grade 8.[8.EE.1]

F-BF.4c (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

F-BF.4d (+) Produce an invertible function from a non-invertible function by restricting the domain.



• In general, $\sqrt{(x-3)^2} = |x-3|$. If we had chosen the dotted portion of the graph, it would have simplified to $-(x-3)$.

F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems Distinguishing between situations that can be modeled with linear functions and with exponential functions[F-LE.1a] turns on understanding their rates of growth and looking for indications of these types of growth rates (MP7). One indicator of these growth rates is differences over equal intervals, given, for example, in a table of values drawn from the situation—with the understanding that such a table may only approximate the situation (MP4).

F-LE.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

To prove that a linear function grows by equal differences over equal intervals,[F-LE.1b] students draw on the understanding developed in Grade 8 that the ratio of the rise and run for any two distinct points on a line is the same (see the Expressions and Equations Progression). An interval can be seen as determining two points on the line whose x-coordinates occur at the boundaries of the intervals. The equal intervals can be seen as the runs for two pairs of points. Because these runs have equal length and ratio between rise and run is the same for any pair of points, the consequence that the corresponding rises are the same. These rises are the growth of the function over each interval.

F-LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

Students note the correspondence between rise and run on a graph and differences of inputs and outputs in a symbolic form of the proof (MP1). A symbolic proof has the advantage that the analogous proof showing that exponential functions grow by equal factors over equal intervals begins in an analogous way.

The process of going from linear or exponential functions to tables can go in the opposite direction. Given sufficient information, e.g., a table of values together with information about the type of relationship represented,[F-LE.4], students construct the appropriate function. For example, students might be given the information that the table below shows inputs and outputs of an exponential function, and asked to write an expression for the function.

F-LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Input	Output
0	5
8	33

Interpret expressions for functions in terms of the situation they model Students may build a function to model a situation, using parameters from that situation. In these cases, interpreting expressions for a linear or exponential function in terms of the situation it models[F-LE.5] is often just a matter of remembering how the function was constructed. However, interpreting expressions may be less straightforward for students when they are given an algebraic expression for a function and a description of what the function is intended to model.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

For example, in doing the task Illegal Fish in the margin, students may need to rely on their understanding of a function as determining an output for a given input to answer the question "Find b if you know the lake contains 33 fish after eight weeks."

See the linear and exponential model section of the Modeling Progression for an example of an interpretation of the intersection of a linear and an exponential function in terms of the situation they model.

Illegal Fish

A fisherman illegally introduces some ïňĄsh into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where x is the time in weeks following the introduction and b is a positive unknown base.

(a) Exactly how many fish did the fisherman release into the lake?

(b) Find b if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

(c) Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/579

Trigonometric Functions

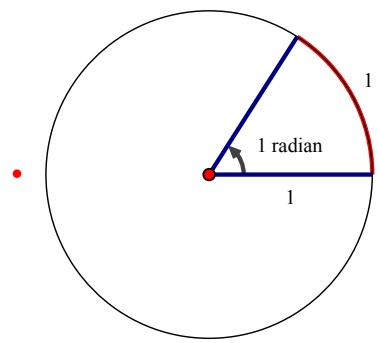
Right triangle trigonometry is concerned with ratios of sides of right triangles, allowing functions of angle measures to be defined in terms of these ratios.• This limits the angles considered to those between $0°$ and $90°$. Circular trigonometry extends the domains of the trigonometric functions within the real numbers.

Traditionally, trigonometry includes six functions (sine, cosine, tangent, cotangent, secant, cosecant). Because the second three may be expressed as reciprocals of the first three, this progression discusses only the first three.

• Traditionally, trigonometry concerns "ratios." Note, however, that according to the usage of the Ratio and Proportional Reasoning Progression, that these would be called the "value of the ratio." In high school, students' understanding of ratio may now be sophisticated enough to allow "ratio" to be used for "value of the ratio" in the traditional manner. Likewise, angles are carefully distinguished from their measurements when students are learning about measuring angles in Grades 4 and 5. In high school, students' understanding of angle measure may now allow angles to be referred to by their measures.

Extend the domain of trigonometric functions using the unit circle In circular trigonometry, angles are usually measured in radians rather than degrees. Radian measure is defined so that in the unit circle the measure of an angle is equal to the length of the intercepted arc.[F-TF.1] An angle of measure 1 radian turns out to be approximately $57.3°$.• A full revolution, which corresponds to an angle of $360°$, has measure equal to the circumference of the unit circle, or 2π. A quarter turn, or $90°$, measures $\pi/2$ radians.

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.



In geometry, students learn, by similarity, that the radian measure of an angle can be defined as the quotient of arc length to radius.[G-C.5] As a quotient of two lengths, therefore, radian measure is "dimensionless." That is why the "unit" is often omitted when measuring angles in radians.

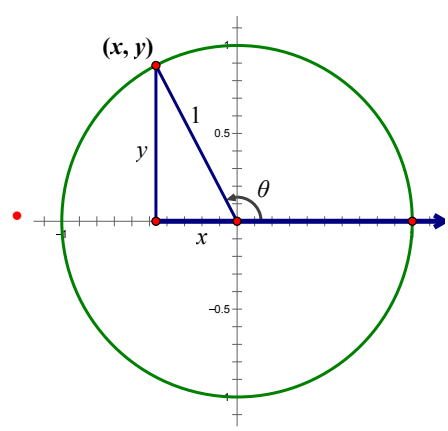
G-C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

In calculus, the benefits of radian measure become plentiful, leading, for example, to simple formulas for derivatives and integrals of trigonometric functions. Before calculus, there are two key benefits of using radians rather than degrees:

- arclength is simply $r\theta$, and
- $\sin\theta \approx \theta$ for small θ.

In right triangle trigonometry, angles must be between $0°$ and $90°$. Circular trigonometry allows angles that describe any amount of rotation, including rotation greater than $360°$.[F-TF.2] Consistent with conventions for measuring angles, counterclockwise rotation is associated with angles of positive measure and clockwise rotation is associated with angles of negative measure. The topic is called circular trigonometry because all rotations take place about the origin on the circle of radius 1 centered at the origin (called the unit circle), in the the coordinate plane.

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.



With the help of a picture,• students mark the intended angle, θ, in radians, measured counterclockwise from the positive ray of the x-axis; identify the coordinates x and y; draw a reference triangle; and then use right triangle trigonometry. In particular, $\sin\theta = y/1 = y$, $\cos\theta = x/1 = x$, and $\tan\theta = y/x$. (Note the simplicity afforded by using a circle of radius 1.) This way, students can compute values of any of the trigonometric functions, except that in circular

trigonometry being careful to note the signs of x and y. In the figure as drawn in the second quadrant, for example, x is negative and y is positive, which implies that $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are both negative.

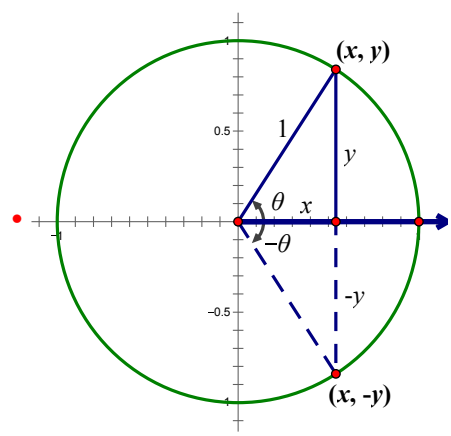
The next step is sometimes called "unwrapping the unit circle." On a fresh set of axes, the angle θ is plotted along the horizontal axis and one of the trigonometric functions is plotted along the vertical axis. Dynamic presentations with shadows can help considerably, and the point should be that students notice the periodicity of the functions, caused by the repeated rotation about the origin, regularly reflecting on the grounding in right triangle trigonometry.

With the help of the special right triangles, $30°$-$60°$-$90°$ and $45°$-$45°$-$90°$, for which the quotients of sides are easily computed, the values of the trigonometric functions are easily computed for the angles $\pi/3$, $\pi/4$, and $\pi/6$ as well as their multiples.[F-TF.3] For advanced mathematics, students need to develop fluency with the trigonometric functions of these special angles to support fluency with the "unwrapping of the unit circle" to create and graph the trigonometric functions.

Using symmetry,[•] students can see that, compared to the reference triangle with angle θ, an angle of $-\theta$ will produce a congruent reference triangle that is its reflection across the x-axis. They can then reason that $\sin(-\theta) = -y = -\sin(\theta)$, so sine is an odd function. Similarly, $\cos(-\theta) = x = \cos(\theta)$, so cosine is an even function.[F-TF.4] Some additional work is required to verify that these relationships hold for values of θ outside the first quadrant.

The same sorts of pictures can be used to argue that the trigonometric functions are periodic. For example, for any integer n, $\sin(\theta + 2n\pi) = \sin(\theta)$ because angles that differ by a multiple of 2π have the same terminal side and thus the same coordinates x and y.

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.



F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions Now that students are armed with trigonometric functions, they can model some periodic phenomena that occur in the real world. For students who do not continue into advanced mathematics, this is the culmination of their study of trigonometric functions.

The tangent and function is not often useful for modeling periodic phenomena because $\tan x$ is undefined for $x = \frac{\pi}{2} + k\pi$, where k is an integer. Because the sine and cosine functions have the same shape, either suffices to model simple periodic phenomena.[F-TF.5] Functions are called *sinusoidal* if they have the same shape as the sine graph. Many real-world phenomena can be approximated by sinusoids, including sound waves, oscillation on a spring, the motion of a pendulum, tides, and phases of the moon. Some students will learn in college that sinusoids are used as building blocks to approximate any periodic waveform.

The general sinusoid is given by $f(t) = A + B\sin(Ct + D)$. Students can reason that because $\sin(\theta)$ oscillates between -1 and 1,

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

$A+B\sin(Ct+D)$ will oscillate between $A-B$ and $A+B$. Thus, $y = A$ is the midline, and B is the amplitude of the sinusoid. Students can obtain the frequency from this equation by noting the period of $sint$ of 2π and knowing the frequency is the reciprocal of the period, so the period of $sinCt$ is $2\pi/C$. When modeling, students need to have the sense that C affects the frequency and that C and D together produce a phase shift, but getting these correct might involve technological support, except in simple cases.

For example, students might be asked to model the length of the day in Columbus, Ohio. Day length as a function of date is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference. So $A \approx 12.17$, and $B \approx 2.83$. With some support, students can determine that for the period to be 365 days (per cycle),• $C = 2\pi/365$, and if day 0 corresponds to March 21, no phase shift is needed, so $D = 0$. Thus,

$$f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$$

From the graph,• students can see that the period is indeed 365 days, as desired, as it takes one year to complete the cycle. They can also see that two days are approximately 14 hours long, which is to say that $f(t) = 14$ has two solutions over a domain of one year, and they might use graphing or spreadsheet technology to determine that May 1 and August 10 are the closest such days. Students can also see that $f(t) = 9$ has no solutions, which makes sense because 9 hours, 20 minutes is the minimum length of day.

+ Students who take advanced mathematics will need additional fluency with transformations of trigonometric functions, including changes in frequency and phase shifts.

+ Based on plenty of experience solving equations of the form $f(t) = c$ graphically, students of advanced mathematics should be able to see that such equations will have an infinite number of solutions when f is a trigonometric function. Furthermore, they should have had experience of restricting the domain of a function so that it has an inverse. For trigonometric functions, a common approach to restricting the domain is to choose an interval on which the function is always increasing or always decreasing.[F-TF.6] The obvious choice for $\sin(x)$ is the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, shown as the solid part of the graph.• This yields a function $\theta = \sin^{-1}(x)$ with domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

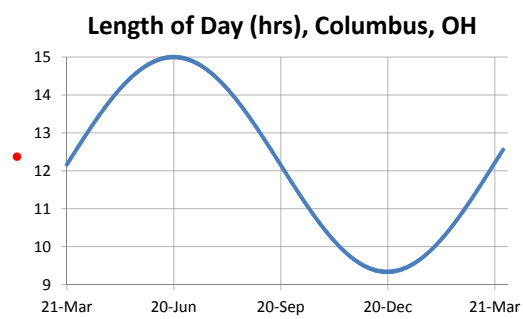
Inverses of trigonometric functions can be used in solving equations in modeling contexts.[F-TF.7] For example, in the length of day context, students can use inverse trig functions to determine days with particular lengths. This amounts to solving $f(t) = d$ for t, which yields

$$t = \frac{365}{2\pi}\sin^{-1}\left(\frac{d - 12.17}{2.83}\right)$$

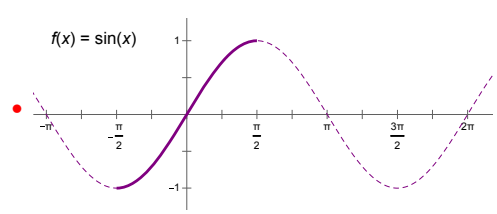
Frequency vs. Period

For a sinusoid, the frequency is often measured in cycles per second, thus the period is often measured in seconds per cycle. For reasoning about a context, it is common to choose whichever is greater numerically.

• or for the frequency to be $\frac{1}{365}$ cycles/day



F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.



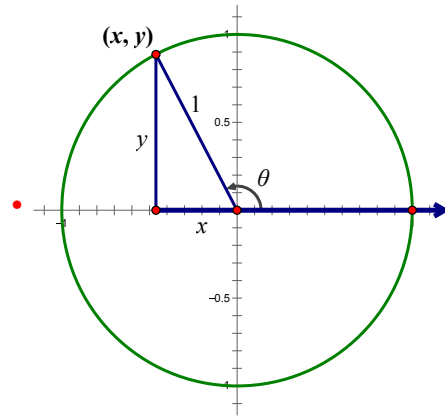
F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

+ Using $d = 14$ and a calculator (in radian mode), they can compute
+ that $t \approx 40.85$, which is closest to May 1. Finding the other solution
+ is a bit of a challenge, but the graph indicates that it should occur
+ just as many days before midyear (day 182.5) as May 1 occurs after
+ day 0. So the other solution is $t \approx 182.5 - 40.85 = 141.65$, which is
+ closest to August 10.

Prove and apply trigonometric identities For the cases illustrated by the diagram•(in which the terminal side of angle θ does not lie on an axis) and the definitions of $\sin\theta$ and $\cos\theta$, students can reason that, in any quadrant, the lengths of the legs of the reference triangle are $|x|$ and $|y|$. It then follows from the Pythagorean Theorem that $|x|^2 + |y|^2 = 1$. Because $|a|^2 = a^2$ for any real number a, this equation can be written $x^2 + y^2 = 1$. Because $x = \cos\theta$ and $y = \sin\theta$, the equation can be written as $\sin^2(\theta) + \cos^2(\theta) = 1$. When the terminal side of angle θ does lie on an axis, then one of x or y is 0 and the other is 1 or -1 and the equation still holds. This argument proves what is known as the Pythagorean identity[F-TF.8] because it is essentially a restatement of the Pythagorean Theorem for a right triangle of hypotenuse 1.

With this identity and the value of one of the trigonometric functions for a given angle, students can find the values of the other functions for that angle, as long as they know the quadrant in which the angle lies. For example, if $\sin(\theta) = 0.6$ and θ lies in the second quadrant, then $\cos^2(\theta) = 1 - 0.6^2 = 0.64$, so $\cos(\theta) = \pm\sqrt{0.64} = \pm 0.8$. Because cosine is negative in the second quadrant, it follows that $\cos(\theta) = 0.8$, and therefore $\tan(\theta) = \sin(\theta)/\cos(\theta) = 0.6/(-0.8) = -0.75$.

+ Students in advanced mathematics courses can prove and use
+ other trigonometric identities, including the addition and subtrac-
+ tion formulas.[F-TF.9] If students have already represented complex
+ numbers on the complex plane[N-CN.4] and developed the geomet-
+ ric interpretation of their multiplication,[N-CN.5] then the the product
+ $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ can be used in deriving the addition
+ formulas for cosine and sine. Subtraction and double angle formulas
+ can follow from these.



F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

F-TF.9(+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

N-CN.4(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5(+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.